Lie algebras attain perfection and reveal deep unexpected relations
Both Dynkin and Coxeter arrived at a graphical encoding of the root systems. These are systems of vectors in Euclidian space like that shown here on the right which is the $A_3$ system. They had different targets: Dynkin was interested in Lie Algebras, Coxeter in reflection groups. There is a deep relation between the two.
Dynkin’s life started in Sankt Peterburg, already renamed Leningrad, was early marked by the ominous shadow of Stalin’s purges and continued to be difficult and insecure until his final emigration to the United States in 1976. In 1935, he and his family, of Jewish origin, were forcefully expelled from their native town and were exiled to Kazachstan. Two years later, Eugene’s father, Boris, previously a well-to-do lawyer, was arrested without any concrete charge, declared to be a People’s Enemy and executed. Eugene Dynkin succeeded to be admitted to Moscow University in 1940. He said: It was almost a miracle that I was admitted to Moscow University. Every step in my professional career was difficult because the fate of my father, in combination with my Jewish origin. It was thanks to Kolmogorov.

It was during his student years that Dynkin, trying to understand Weyl’s writings on Lie Groups, invented the Dynkin diagrams to classify Cartan matrices. Similar graphs had been independently introduced by Coxeter in his study of reflection groups, presently named Coxeter groups.
What are Dynkin Diagrams?

They are a graphical representation of the Cartan matrix, the matrix of scalar products of the simple roots

\[ C_{ij} = \langle \alpha_i, \alpha_j \rangle \equiv 2 \frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} \]

**Root systems** \( \Delta \) and **simple roots** \( \alpha_i \) are relatively simple concepts that can be illustrated to a public of non-experts.

In the figure, working in \( D=3 \), we display 12 vectors that form a **root system** \( \Delta \). The three blue vectors are the simple roots. The black ones are the other roots that are linear combinations of the simple ones with coefficients that are all positive or all negative integers.
Root system (definition)

(a) Given any pair of vectors $\alpha$ and $\beta$ that belong to the root system and considering scalar products in the ordinary euclidean space $\mathbb{R}^r$, we have that the following ratio $2 \frac{(\alpha, \beta)}{(\alpha, \alpha)}$ must be an integer number, namely:

$$2 \frac{(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z}$$

(b) Given any root $\alpha$ belonging to the root system $\Delta$ and considering the hyperplane $\text{Hyp}_\alpha$ orthogonal to it, the reflection of any root with respect to such an hyperplane is again a root of the same system. This property is illustrated for the planar case $\alpha_2$ in Fig. 5.11 and it is illustrated for the three-dimensional case $\alpha_3$ in Fig. 5.13. Mathematically the described property is formulated as follows:

$$\forall \alpha, \beta \in \Delta : \quad \sigma_\alpha(\beta) \equiv \beta - 2 \alpha \frac{(\alpha, \beta)}{(\alpha, \alpha)} \in \Delta$$
Root system (Theorem)

In every root system $\Delta$ of rank $\ell$ there exists a subset $\alpha_1, \alpha_2, \ldots, \alpha_\ell$ of linearly independent roots, named the *simple roots*, that form a basis for $\mathbb{R}^\ell$ and are such that every other root $\beta = \sum_{i=1}^{\ell} n_i \alpha_i$ is a linear combination of these simple ones with integer valued coefficients $n_i$ that are either all positive or all negative.
triangles with quantized angles

\[ \theta = 30 \; \text{or} \; 45 \; \text{or} \; 60 \; \text{or} \; 90 \; \text{degrees} \]

The angle among two roots in a root system is one of the Platonic angles or one of its multiples:

\[ \theta = 30, \text{or} \; 60, \text{or} \; 45, \text{or} \; 90, \text{or} \; 135, \text{or} \; 120, \text{or} \; 180, \text{or} \; 270 \; \text{degrees} \]

Every root system defines a group formed of reflections the Weyl group. Coxeter studied the groups made of reflections and arrived at the same type of diagrams classified by Dynkin. All Weyl groups are Coxeter, but there are Coxeter groups that are not Weyl.
Donald Coxeter’s father, Harold, was a gas manufacturer while his mother, Lucy, was a painter.

The artistic tendencies of Donald can be probably traced back to his mother’s legacy. Donald was educated at the University of Cambridge, B.S. in 1929. In 1936 Coxeter received an appointment from the University of Toronto in Canada that he accepted. He remained on the faculty at Toronto until his death in 2003. His work was in Geometry. Coxeter polytopes are defined as the fundamental domains of discrete reflection groups, named Coxeter groups, and they give rise to tessellations. In 1934 he classified all spherical and euclidian reflection groups. In this context he introduced Coxeter diagrams. He was fascinated by the work of the Dutch painter Escher whom he met in 1954 building up a life-long friendship, certainly eased by the Dutch nationality of his own wife Rien.

It was the time of Khrushchev’s reforms. The following year, with Kolmogorov’s strong support, Dynkin was appointed to a chair at the University of Moscow and he held this chair until 1968. In 1968, the year of Prague Spring and of its suppression by the Soviet tanks sent by Brezhnev, Dynkin was removed from his chair at Moscow University because he had signed a letter in support of the two dissidents Alexander Ginzburg and Yuri Galanskov who were at that time on trial for compiling the White Book. Dynkin, removed from Moscow State University, was simply sent to the Institute of Central Economics and Mathematics of the USSR Academy of Sciences. He worked there from 1968 to 1976. At the end of 1976, Dynkin left the USSR for the United States. In the United States, Dynkin was offered a chair by Cornell University, which he accepted. He stayed in Ithaca, New York State, the rest of his life until his death in 2014.
The relation with Lie Algebras is

\[
\begin{align*}
[H_i, H_j] &= 0 \\
[H_i, E^\alpha] &= \alpha_i E^\alpha \\
[E^\alpha, E^{-\alpha}] &= \alpha^i H_i \\
[E^\alpha, E^\beta] &= N(\alpha, \beta) E^{\alpha+\beta} \quad \text{if} \quad \alpha + \beta \in \Delta \\
[E^\alpha, E^\beta] &= 0 \quad \text{if} \quad \alpha + \beta \notin \Delta
\end{align*}
\]

Where $\Delta$ is the root system.

Every semisimple Lie algebra can be put into this form that is named:

The Cartan-Weyl form of the Lie Algebra

In Mathematics one of the main issues is always classification:
What are the possible semisimple Lie algebras?
What are the possible root systems?
What are the possible Cartan matrices?

ANSWER:
Classify the Dynkin diagrams
The classification

Classical Lie Group/Algebras

<table>
<thead>
<tr>
<th>$\alpha_\ell$</th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>$\alpha_3$</th>
<th>$\alpha_{\ell-2}$</th>
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<td>$\delta_\ell$</td>
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<td>$\alpha_{\ell-3}$</td>
<td>$\alpha_{\ell-2}$</td>
<td>$\alpha_{\ell-1}$</td>
</tr>
</tbody>
</table>

Exceptional Lie Group/Algebras

| $\varepsilon_6$ | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $\alpha_5$ | $\alpha_6$ | $\alpha_4$ |
| $\varepsilon_7$ | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $\alpha_5$ | $\alpha_6$ | $\alpha_7$ | $\alpha_4$ |
| $\varepsilon_8$ | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $\alpha_5$ | $\alpha_6$ | $\alpha_7$ | $\alpha_8$ | $\alpha_4$ |
| $f_4$          | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $\alpha_4$ |
| $g_2$          | $\alpha_1$ | $\alpha_2$ |

This was the result of Killing revisited and reinforced by Cartan.
A lot of contemporary results in the Superworld are related with....

The interpretation of the role of exceptional Lie Algebras that, up to the Advent of the Superworld, were regarded by physicists as Mathematical Curiosities

This is an important lesson to be learned: sporadic structures always hide some fundamental truth relevant for the explanation of the Universe (at least in our Wester Analytic System of Thought!)
The Connection Army

Development of the Fiber Bundle notion and of the notion of a Connection on a Principal Bundle
A Gauge Field in Physics Parlance
The Generals of the Connection Army

C. N. Yang 1922
Yang Mills paper 1954

Charles Ehresman (1905 - 1979)

Paper on the connection 1950
Slowly the concept of fibre bundle emerges in the thirties and the forties of the XXth century.

The Mobius strip is the prototype of a manifold that is locally a trivial product of two manifolds, yet globally it is not a product. Indeed it is a fibre bundle.
Fibre Bundles in a few pictures

\[ \phi_\alpha : \pi^{-1}(U_\alpha) \subset P \rightarrow U_\alpha \otimes F \]

\[ \pi \circ \phi_\alpha^{-1}(p, f) = p \]

\[ t_{\alpha\beta} \equiv \phi_\beta^{-1} \circ \phi_\alpha : (U_\alpha \cap U_\beta) \otimes F \rightarrow (U_\alpha \cap U_\beta) \otimes F \]
Belong to a Lie Group G (the structural group)
G acts as a group of transformations on the standard fibre F
When F=G is the Lie group itself we have a Principal Bundle
Principal Bundles are the ancestors of an infinite tower of associated vector bundles, one for each linear representation of G.
In Physics, all matter fields are sections of a vector bundle associated to a Principal Bundle.

The concept of section of a fibre-bundle is illustrated by the above picture. To every point $p$ of the base manifold a section $s$ associates, in a continuous way, a point of the total space $s(p) \in P$, that must belong to the fibre over $p$, namely such that $\pi(s(p)) = p$. In the case of vector bundles the section image $s(p)$ of a base manifold point $p$ is necessarily an $r$-dimensional vector, $r$ being the rank of the bundle.

A frame over $U$ is a set of $r$ sections $\{s_1, \ldots, s_r\}$ such that $\{s_1(z), \ldots, s_r(z)\}$ is a basis for $\pi^{-1}(p)$ for any $p \in U$, having denoted by $z^i$ the coordinates labeling the points of the base manifold in the chosen patch.
All Physics boils down to this question: What is vertical, what is horizontal?

Charles Ehresman gave the answer to this question in 1950

The Enchanted Mountain of the Episteme
Charles Ehresman

He was born in German speaking Alsace in 1905 from a poor family. His first education was in German, but after Alsace was returned to France in 1918, as a result of Germany’s defeat in World War I, Ehresman attended only French schools and his University education was entirely French. Indeed in 1924 he entered the École Normale Superieure from which he graduated in 1927.

After that, he served as a teacher of Mathematics in the French colony of Morocco and then he went to Göttingen that in the late twenties and beginning of the thirties was the major scientific center of the world, at least for Mathematics and Physics. The raising of Nazi power in Germany dismantled the scientific leadership of the country, caused the decay of Göttingen and obliged all the Jewish scientists, who so greatly contributed to German culture, to emigrate to the United States. Ehresman also fled from Göttingen to Princeton where he studied for few years until 1934. In that year he returned to France to obtain his doctorate under the supervision of Élie Cartan. Charles Ehresman was professor at the Universities of Strasbourg and Clermont Ferrand. In 1955 a special chair of Topology was created for him at the University of Paris which he occupied up to his retirement in 1975. He died in 1979 in Amiens where his second wife, also a mathematician held a chair.
Ehresman’s Connection

Let \( P(M, G) \) be a principal fibre-bundle. A \textit{connection} on \( P \) is a rule which at any point \( u \in P \) defines a unique splitting of the tangent space \( T_u P \) into the vertical subspace \( V_u P \) and into a horizontal complement \( H_u P \).

The algorithmic way to implement the splitting rule advocated by the Ehresmann definition is provided by introducing a connection one-form \( A \) which is just a Lie algebra valued differential one-form on the bundle \( P \).

\begin{align*}
(i) \quad \forall X \in G : \quad A(X^\#) &= X \\
(ii) \quad \forall g \in G : \quad g^*A &= g^{-1}Ag
\end{align*}

\( H_u P \equiv \{X \in T_u P \mid A(X) = 0\} \)

\[ A = g \cdot \mathcal{A} \cdot g^{-1} + dg \cdot g^{-1} \]
Conservation of Isotopic Spin and Isotopic Gauge Invariance*

C. N. Yang † and R. L. Mills
Brookhaven National Laboratory, Upton, New York
(Received June 28, 1954)

It is pointed out that the usual principle of invariance under isotopic spin rotation is not consistent with the concept of localized fields. The possibility is explored of having invariance under local isotopic spin rotations. This leads to formulating a principle of isotopic gauge invariance and the existence of a b field which has the same relation to the isotopic spin that the electromagnetic field has to the electric charge. The b field satisfies nonlinear differential equations. The quanta of the b field are particles with spin unity, isotopic spin unity, and electric charge ±e or zero.

\[ B'_\mu = S^{-1}B_\mu S + \frac{i}{\epsilon} \frac{\partial S}{\partial x_\mu} \]

The connection one-form of Ehresman connection is the gauge field of Yang-Mills theory

\[ A = B_\mu dx^\mu \]
Historical Summary on Gauge Fields

- Strictly speaking the very first to introduce a connection was Christoffel, whose paper on the coefficients named after him dates to 1869. *Levi Civita connection* = gravity.
- Almost immediately after him, in 1873 Maxwell was the second to introduce a connection, this time on a U(1) principal bundle = electromagnetism.
- In 1923 Cartan formalized the notion of affine connections.
- *Ehresman connection* on a principal bundle was introduced in 1950.
- As early as 1929, Hermann Weyl had introduced his peculiar gauge theory based on scale transformations rather than phase transformations, as it is appropriate for electromagnetism.
Biography of C.N. Yang

Yang was born in Hefei, Anhui, China; his father, Yang Wuzhi (1896–1973), was a mathematician, and his mother, Luo Meng-hua, was a housewife. Yang attended elementary school and high school in Beijing, and in the autumn of 1937 his family moved to Hefei after the Japanese invaded China. In 1944 he received his master's degree from Tsinghua University, which had moved to Kunming during the Sino-Japanese War (1937–1945). From 1946, Yang studied with Edward Teller (1908–2003) at the University of Chicago, where he received his doctorate in 1948. He remained there for a year as an assistant to Enrico Fermi. In 1949 he was invited to do his research at the Institute for Advanced Studies in Princeton. He was made a permanent member of the Institute in 1952, and full professor in 1955. In 1965 he moved to Stony Brook University, where he was named the Albert Einstein Professor of Physics and the first director of the newly founded Institute for Theoretical Physics. After retirement from Stony Brook, Yang now resides in China, and he was granted permanent residency in China in 2004. He renounced his U.S. citizenship in 2015 and reclaimed his Chinese citizenship.

Nobel Prize in Physics (1957)
Some Philosophical Remarks

From this short summary not only we can fully appreciate the meaning of Yang’s picture but we also learn another important lesson. Observing the history of science on a longer time-scale we see that the Galilean Method consisting of the three phases:

- (a) Interrogation of Nature
- (b) Formulation of a Theory to explain Observed Phenomena
- (c) Verification or Falsification of the further predictions of the Theory

is very important and valuable but it is not the end of the story.

Indeed there is not only Nature that has to be interrogated, but also Abstract Human Thought which finds its most efficient way of expression in the language of Mathematics.

There exists, historically, an independent logical development of mathematical notions and constructions, whose point of origin is of philosophical nature, rooted in a System of Thought which is civilization dependent. Fundamental steps forward in physics occur quite often through a process ofagnition: an existing mathematical structure is recognized to be the category encompassing fundamental concepts elaborated in physics. At that moment all the conceptual implications for physical thought of that mathematical structure are activated and a new vision emerges.
How Global Properties came to prominence
The main question was: Given a manifold M and a structural group G, can I classify all the principal bundles P(M,G)? The answer was provided by the theory of characteristic classes developed by Chern and Weyl. It relies on the ideas of homology and cohomology developed after Poincaré by Hodge, de Rham and others.
All the curves A, B, C are closed because they neither have a beginning nor an ending. Indeed they are loops. There is a difference between A, B and C. If you cut the surface along A or B, it does not split in two parts. Hence neither A, nor B are the boundary of a region. If you cut the surface along C it splits in two parts $R_1$ and $R_2$. Hence C is the boundary of these two regions. Every boundary is closed but not all curves (or surfaces) are boundaries. THIS IS HOMOLOGY in a nut-shell.
Excursus on Cohomology

\[ \ker \partial_i \supset \text{Im} \partial_{i-1} \]

Let us begin with Fig. 8.12. The fundamental idea underlying cohomology theory is captured by that image. There is a sequence of spaces \( \Omega^{[i]} \), whose elements we name the \textit{cochains} and there is a linear operator, named \( d \) (the exterior derivative) that provides \textit{non surjective maps} from each space \( \Omega^{[i]} \) to the next one \( \Omega^{[i+1]} \):

\[ \partial_i : \Omega^{[i]} \xrightarrow{d} \Omega^{[i+1]} ; \quad \forall \phi \in \Omega^{[i]} \quad d\phi \in \Omega^{[i+1]} \]

The fundamental property of the operator \( d \) is its nilpotency, namely it squares to zero \( d^2 = 0 \). In practice this means that the kernel of the map \( \partial_i \), whose elements we name the \textit{cocycles} always contains the image \( \text{Im} \partial_{i-1} \) of the previous map \( \partial_{i-1} \), namely the subspace of \( \Omega^{[i]} \) formed by all those elements that can be written as \( d\phi \) for some \( \phi \) belonging to \( \Omega^{[i-1]} \). We name \textit{coboundaries} the elements of \( \text{Im} \partial_{i-1} \).
Homology or Cohomology classes are equivalence classes

Such a scenario occurs in various mathematical constructions and it is named an elliptic complex $\mathcal{C}$. The cohomology groups of the complex, usually denoted $H^i(\mathcal{C})$ are defined as the set of equivalence classes in which the subspace $\ker \partial_i$ is partitioned with respect to the following equivalence relation:

$$\forall \omega^i, \psi^i \in \ker \partial_i : \omega^i \sim \psi^i \iff (\omega^i - \psi^i) \in \Im \partial_{i-1} \quad (8.2.31)$$
In Paris Elie Cartan’s mathematical ideas provide powerful inspiration for all, worldwide. Emile Picard, André Weil, Georges de Rham studies in Lausanne but he writes his Ph.D thesis in Paris under Cartan’s supervision. Because of the rise of Nazism the world center of Mathematics and Physics shifts from Göttingen to Princeton. In Göttingen Hilbert retires, yet until 1933 Weyl is still there: Ehresman, de Rham, Kolmogorov and other cross through the town. Sing-shen Chern from Peking goes to Hamburg and then Paris, eventually to Princeton. In Hamburg the young Kähler meets with the young Chern. Hodge works in Cambridge.
Chern

Born in China, son to a classical Confucian scholar, Chern studied there mathematics until 1932, when he left for Europe, ending up first in Hamburg and then in Paris. In Hamburg he met with Kähler who introduced him to Cartan’s works. He graduated from Hamburg University in 1936. The same year he reached Paris where he interacted with Cartan himself and met with André Weil. In 1937 he left Paris and went back to China. He took a professorship of mathematics at the Tsing Hua University. In Paris he was strongly marked by Cartan’s influence. ABOUT CARTAN HE WROTE:

There is a tendency in mathematics to be abstract and have everything defined, whereas Cartan approached mathematics more intuitively. That is, he approached mathematics from evidence and the phenomena which arise from special cases rather than from a general and abstract viewpoint.

Usually the day after meeting with Cartan I would get a letter from him. He would say, After you left, I thought more about your questions ... - he had some results, and some more questions, and so on. He knew all these papers on simple Lie groups, Lie algebras, all by heart. When you saw him on the street, when a certain issue would come up, he would pull out some old envelope and write something and give you the answer. And sometimes it took me hours or even days to get the same answer. I saw him about once every two weeks, and clearly I had to work very hard.
Born in Paris in a Jewish family that had escaped from Alsace after the 1870 annexation to the German Empire, he graduated there in 1928 under the supervision of Hadamard. Interested in classical languages and ancient cultures he had an experience as a teacher at a Muslim University in India. Back to France, he interacted with Cartan’s son Henri in Strasbourg. He was member of the Bourbaki group. In 1939 he was in Finland when the Finnish-Soviet war broke up. He was arrested as a spy. Released, he went back to France to be arrested once again as a renitent to the military service. Released, he took part in the 1940 campaign and in the May Debâcle. In January 1941 he fled from France to the US with his entire family. During the war he survived teaching in a minor university in Pennsylvania.

When Chern arrived in Princeton in 1943, after his adventurous air-trip across three continents, he found there such outstanding mathematical personalities as Hermann Weyl, Claude Chevalley and Solomon Lefshetz. Einstein was also in the group. Without any doubt in such an environment Chern’s ideas found a fertile humus where to grow. However he had not forgotten André Weil with whom he had met seven years before in Paris and who was teaching in Pennsylvania, few tens of miles away from Princeton.
The two met several times, talking about Cartan’s mathematics and certainly such conversations are responsible for their almost simultaneous but independent discovery of the Chern–Weil homomorphism... WEIL WROTE

...we seemed to share a common attitude towards such subjects, or towards mathematics in general; we were both striving to strike at the root of each question while freeing our minds from preconceived notions about what others might have regarded as the right or the wrong way of dealing with it.

Given the possible principal bundles $P(M,G)$ on the base manifold $M$ and structural group $G$, the Chern-Weil homomorphism is between the algebra of $G$-invariant polynomials on the Lie Algebra $G$ and the de Rham cohomology ring of the manifold $M$. 

$$\mathcal{C}(G)^G \rightarrow H^*(\mathcal{M},\mathbb{R})$$
The possible principal bundles are classified by the characteristic classes. These are closed differential forms on the base-manifold $M$. If $M$ is topologically trivial also the principal bundles over it are trivial. We are going to analyze the other fundamental isomorphism proved by de Rham between singular homology and cohomology. Indeed the invariant polynomials mentioned in the homomorphism are polynomials in the curvature $F$ of any connection $A$.

\[
\begin{align*}
F &= dA + A \wedge A \\
\mathbb{C}(G)^G \ni \Psi(F) &= \text{Tr}(F \wedge F \cdots \wedge F) + \ldots
\end{align*}
\]

Hence the theory of fibre-bundles and characteristic classes was developed in those years while Ehresman definitely fixed the notion of a connection on a Principal Bundle. The appropriate mathematical language of modern gauge theories in which the Standard Model of non gravitational interactions could be properly formulated was essentially ready by the mid fifties of the XXth century. Yet it took half a century before theoretical physicists became fully aware of the mathematical nature of those objects with which they were playing.
Georges de Rham

Born in 1903 in the small town of Roche at the foot of the Jura, de Rham made his first studies there. In 1919 his family moved to the city of Lausanne, where Georges lived most of his life up to his death in 1990. Since 1924 he attended the University of Lausanne but in 1931 he wrote his doctoral thesis Sur l'Analysis situs des variétés à plusieurs dimensions under the supervision of Élie Cartan and defended it in Paris, obtaining his doctorate in mathematical sciences from the Faculty of Science, University of Paris.

His greatest achievement in mathematics is the theorem that bears his name and states the isomorphism of de Rham cohomology of differential forms with the dual of singular homology based on simplexes.
Every p-simplex has a well defined boundary that is a collection of p+1 faces that are (p-1)-simplexes. Any manifold $M$ can be decomposed into simplexes and in this way one can construct its boundary $\partial M$.

**Definition** << Let us consider the Euclidean space $\mathbb{R}^{p+1}$. The standard p-simplex $\Delta^p$ is the set of all points $\{t_0, t_1, \ldots, t_p\} \in \mathbb{R}^{p+1}$ such that the following conditions are satisfied:

$$t_i \geq 0 \quad ; \quad t_0 + t_1 + \cdots + t_p = 1$$

>>

1-symplex = segment

2-symplex = equilateral triangle

3-symplex = regular tetrahedron
Homology is the elliptic complex where the chains are the $p$-dimensional submanifolds and the $d$-operator is the boundary. The boundary of a boundary vanishes. Cycles are the submanifolds without boundary. Homology classes are cycles modulo boundaries.
Bott said:

In some sense the famous theorem that bears de Rham name dominated his mathematical life, as indeed it dominates so much of the mathematical life of this whole century.

When I met de Rham in 1949 at the Institute in Princeton he was lecturing on the Hodge theory in the context of his currents. These are the natural extensions to manifolds of the distributions which had been introduced a few years earlier by Laurent Schwartz and of course it is only in this extended setting that both the de Rham theorem and the Hodge theory become especially complete.

The original theorem of de Rham was most probably believed to be true by Poincaré and was certainly conjectured (and even used!) in 1928 by Élie Cartan. But in 1931 de Rham set out to give a rigorous proof. The technical problems were considerable at the time, as both the general theory of manifolds and the singular theory were in their early formative stages.

In a nut-shell the Rings of Homology of a manifold M and of Cohomology (differential forms) of the same are isomorphic.

The isomorphism is realized by integrating closed differential forms on cycles.
Born in Scotland, William Hodge, studied Mathematics first in Edinburgh, then in Cambridge and graduated under the supervision of Edmund Whittaker. In 1936 he was appointed as Lowndean Professor of Astronomy and Geometry in Cambridge, a position that he held up to retirement. Hodge has given outstanding contributions in the field of differential forms, harmonic integrals and complex analysis. He invented Hodge duality. He was knighted by Queen Elizabeth in 1959.

Hodge published a paper on the *Theory of Harmonic Integrals* that won him the 1937 Adams Prize and then, in 1941 the book *The Theory and applications of harmonic integrals* whose content was described by Hermann Weyl as: 

...one of the great landmarks in the history of science in the present century.

Essentially harmonic integrals are the integrals of harmonic forms on cycles. They correspond to the conserved quantities of Physics, like electric charge and its generalizations